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Nested Directed BIB Designs with Block Size Three or Four

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SUMMARY

A directed BIB design DB $(k, \lambda; v)$ is a BIB design B $(k, 2\lambda; v)$ in which each ordered pair of treatments occurs together in exactly λ blocks. A nested directed BIB design NDB $(k, \lambda; v)$ of form $\Pi_n (n^{j_n}, \lambda_n)^{i_n}$, $2 \le n \le k-1$, is a DB $(k, \lambda; v)$ where each block contains $\Sigma_n i_n j_n$ mutually disjoint sub-blocks, $i_n j_n$ of which being partitioned into i_n mutually disjoint families of j_n sub-blocks of size n, the j_n sub-blocks of size n belonging to one distinguished system which forms the collection of blocks of a DB $(n, \lambda_n; v)$. It is shown that the necessary conditions for the existence of such designs with block size 3 or 4 are also sufficient except possibly for an NDB (4, 2; 10) of form $(3, 1)^1$.

Key words : BIB design, Directed BIB design, Nested BIB design, Nested directed BIB design, Nested directed GD design, PBD-closed set.

1. *Introduction*

A balanced incomplete block (BIB) design B $(k, \lambda; v)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a set of v treatments, $\mathcal{B}$ is a collection of k-subsets, called blocks, of $\mathcal{V}$, such that every pair of distinct treatments of $\mathcal{V}$ occurs in exactly λ blocks of $\mathcal{B}$.

A directed BIB design DB $(k, \lambda; v)$ is a B $(k, 2\lambda; v)$ in which the blocks are regarded as ordered k-tuples and in which each ordered pair of distinct treatments occurs in exactly λ blocks. A pair $\{a, b\}$ is said to occur in a block if a is written to the left of b. Such a design is a block design on a directed graph. It may also be regarded as a generalization of crossover designs (see Street and Wilson [12]).

Hung and Mendelsohn [4] first introduced the concept of directed BIB designs. These designs were further discussed by Seberry and Skillicorn [9], Street and Seberry [11] and Street and Wilson [12]. They showed that the necessary conditions for the existence of a DB $(k, \lambda; v)$ are also sufficient for $k = 3, 4$ and 5.

On the other hand, Federer [3] and Preece [8] introduced the concepts of nested BIB designs in different ways for those statistical situations where there are more sources of variation than can be eliminated by ordinary block designs. These designs were further investigated by Colbourn and Colbourn [2], Stinson [10] and others. Recently, these concepts were unified by Kageyama and Miao [6], as follows.

A nested BIB design NB $(k, \lambda; v)$ of form $\Pi_n (n^{j_n}, \lambda_n)^{i_n}$, $2 \le n \le k-1$, is a $B(k, \lambda; v)(\mathcal{V}, \mathcal{B})$ where each block contains $\Sigma_n i_n j_n$ mutually disjoint sub blocks, $i_n j_n$ of which being partitioned into i_n mutually disjoint families of j_n sub-blocks of size n, the j_n sub-blocks of size n belonging to one distinguished system $\mathcal{B}_n(l), 1 \le l \le i_n$, such that $(\mathcal{V}, \mathcal{B}_n(l))$ forms a $B(n, \lambda_n; v)$ for each integer n with $i_n \ge 1$.

Together with some results obtained by others, Kegeyama and Miao ([5], [7]) showed that the necessary conditions for the existence of an NB $(k, \lambda; v)$ of any possible form are also sufficient for $k = 3, 4$ and 5.

On account of all of these observations the natural and direct generalization is the existence problem of nested directed BIB designs, where the concept of nested directed BIB designs can be defined in the following way.

A nested directed BIB design NDB $(k, \lambda; v)$ of form $\Pi_n (n^{j_n}, \lambda_n)^{i_n}$, $2 \le n \le k-1$, is an NB $(k, 2\lambda; v)$ of form $\Pi_n (n^{j_n}, 2\lambda_n)^{i_n}$, $2 \le n \le k-1$, in which the blocks and the sub-blocks are regarded as ordered k-tuples and n-tuples and in which each ordered pair of distinct treatments occurs in exactly λ blocks and λ_n sub-blocks of size n for each integer n with $i_n \ge 1$.

Nested directed BIB designs form a useful class of experimental designs for statistical situations where there are more sources of variability and one source is nested within the other.

By the usual counting arguments for nested BIB designs and the fact that a DB $(k, \lambda; v)$ is in fact a $B(k, 2\lambda; v)$, we can obtain the following.

Theorem 1.1. The necessary conditions for the existence of an NDB $(k, \lambda; v)$ of form $\Pi_n (n^{j_n}, \lambda_n)^{i_n}$ are that

$$\lambda = k(k-1)\frac{\lambda_n}{n(n-1)j_n}, \quad 2\lambda(v-1) \equiv 0 \bmod (k-1)$$

$$2\lambda v(v-1) \equiv 0 \bmod k(k-1), \quad 2\lambda_n (v-1) \equiv 0 \bmod (n-1)$$

$$2\lambda_n v(v-1) \equiv 0 \bmod n(n-1)$$

for all integers n with $i_n \geq 1$.

By using a recursive construction of Wilson type, some direct constructions by difference techniques, and other special constructions, we shall prove in this paper that the necessary conditions as described in Theorem 1.1 are also sufficient for the existence of an NDB $(k, \lambda; v)$ of any possible form when $k = 3$ and 4 with at most one possible exception of an NDB (4, 2; 10) of form $(3, 1)^1$.

2. *A Recursive Construction*

We here describe a recursive construction for nested directed group divisible designs. Intermediate designs will be used. Some definitions are now given.

A group divisible (GD) design GD $(K, \lambda, M; v)$ is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a set of v treatments, $\mathcal{G}$ and $\mathcal{B}$ are collections of some subsets of $\mathcal{V}$, called groups and blocks, respectively, such that

(i) $|G| \in M$ for every $G \in \mathcal{G}$, where $\mathcal{G}$ forms a partition of $\mathcal{V}$,

(ii) $|B| \in K$ for every $B \in \mathcal{B}$,

(iii) $|G \cap B| \leq 1$ for every $G \in \mathcal{G}$ and every $B \in \mathcal{B}$, and

(iv) every pair of treatments $\{x, y\}$, where x and y belong to distinct groups, is contained in exactly λ blocks of $\mathcal{B}$.

The type of a GD design $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. An "exponential" notation is usually used to describe types : a type $g_1^{u_1} g_2^{u_2} \cdots g_m^{u_m}$ denotes u_i occurrences of g_i, $1 \leq i \leq m$.

A directed GD design DGD $(K, \lambda, M; v)$ of type T is a GD $(K, 2\lambda, M; v)$ of the same type T in which each ordered pair of treatments formed from different groups occurs in exactly λ blocks.

A nested directed GD design NDGD $(k, \lambda, M; v)$ $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ of type T and of form $\Pi_n (n^{j_n}, \lambda_n)^{i_n}$, $2 \leq n \leq k-1$, is a DGD $(\{k\}, \lambda, M; v)$ of type T where each block has $\Sigma_n i_n j_n$ mutually disjoint sub-blocks, $i_n j_n$ of which being partitioned into i_n mutually disjoint families of j_n sub-blocks of size n, the j_n sub-blocks of size n belonging to one distinguished system $\mathcal{B}_n(\ell)$, $1 \leq \ell \leq i_n$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B}_n(\ell))$ forms a DGD $(\{n\}, \lambda_n, M; v)$ of type T for each integer n with $i_n \geq 1$ and each integer ℓ with $1 \leq \ell \leq i_n$.

Note that a BIB design $B(k, \lambda; v)$ is a $GD(\{k\}, \lambda, \{1\}; v)$ of type 1^v, a directed BIB design $DB(k, \lambda; v)$ is a $DGD(\{k\}, \lambda, \{1\}; v)$ of type 1^v, and a nested directed BIB design $NDB(k, \lambda; v)$ of form F is an $NDGD(k, \lambda, \{1\}; v)$ of type 1^v and of form F.

For further discussion we need a well-known construction for GD designs due to Wilson [13].

Theorem 2.1. Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a GD design with index λ. Further let $\omega : \mathcal{V} \to \mathcal{N} \cup \{0\}$ be a weight function where $\mathcal{N}$ is the set of all positive integers. For each $B \in \mathcal{B}$, suppose there exists a $GD(K, \lambda', \{\omega(x) : x \in B\}; \Sigma_{x \in B}\, \omega(x))$ of type $\{\omega(x) : x \in B\}$ $(\cup_{x \in B} S(x), \{S(x) : x \in B\}, \mathcal{B}(B))$, where $S(x) = \{x_1, \ldots, x_{w(x)}\}$ for every $x \in \mathcal{V}$. Then there exists a $GD(K, \lambda\lambda', \{\Sigma_{x \in G}\, \omega(x) : G \in \mathcal{G}\}; \Sigma_{x \in \mathcal{V}}\, \omega(x))$ of type $\{\Sigma_{x \in G}\, \omega(x) : G \in \mathcal{G}\}$, $(\cup_{x \in \mathcal{V}} S(x), \{\cup_{x \in G} S(x) : G \in \mathcal{G}\}, \cup_{B \in \mathcal{B}} \mathcal{B}(B))$.

As a variation, we have the following.

Theorem 2.2. Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a GD design with index λ. Further let $\omega : \mathcal{V} \longrightarrow \mathcal{N} \cup \{0\}$ be a weight function. For each $B \in \mathcal{B}$, suppose there exists a $DGD(K, \lambda', \{\omega(x) : x \in B\}; \Sigma_{x \in B}\, \omega(x))$ of type $\{\omega(x) : x \in B\}$ $(\cup_{x \in B} S(x), \{S(x) : x \in B\}, \mathcal{B}(B))$, where $S(x) = \{x_1, \ldots, x_{\omega(x)}\}$ for every $x \in \mathcal{V}$. Then there exists a $DGD(K, \lambda\lambda', \{\Sigma_{x \in G}\, \omega(x) : G \in \mathcal{G}\}; \Sigma_{x \in \mathcal{V}}\, \omega(x))$ of type $\{\Sigma_{x \in G}\, \omega(x) : G \in \mathcal{G}\}$, $\cup_{x \in \mathcal{V}} S(x)$, $\{\cup_{x \in G} S(x) : G \in \mathcal{G}\}$ $\cup_{B \in \mathcal{B}} \mathcal{B}(B))$.

This theorem can be used to give the present recursive construction for nested directed GD designs.

Theorem 2.3. Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a GD design with index λ. Further let $\omega : \mathcal{V} \to \mathcal{N} \cup \{0\}$ be a weight function. For each $B \in \mathcal{B}$, suppose there exists an $NDGD(k, \lambda', \{\omega(x) : x \in B\}; \Sigma_{x \in B}\, \omega(x))$ of type $\{\omega(x) : x \in B\}$ and of form $\Pi_n (n^{j_n}, \lambda_n)^{i_n}$, $(\cup_{x \in B} S(x), \{S(x) : x \in B\}, \mathcal{B}(B))$, where $S(x) = \{x_1, \ldots, x_{\omega(x)}\}$ for every $x \in \mathcal{V}$. Then there exists an $NDGD(k, \lambda\lambda', \{\Sigma_{x \in G}\, \omega(x) : G \in \mathcal{G}\}; \Sigma_{x \in \mathcal{V}}\, \omega(x))$ of type $\{\Sigma_{x \in G}\, \omega(x) : G \in \mathcal{G}\}$ and of form $\Pi_n (n^{j_n}, \lambda\lambda_n)^{i_n}$, $(\cup_{x \in \mathcal{V}} S(x), \{\cup_{x \in G} S(x) : G \in \mathcal{G}\}, \cup_{B \in \mathcal{B}} \mathcal{B}(B))$.

As an immediate consequence, we can obtain the following structural property of nested directed BIB designs, which all be utilized later. A pairwise balanced (PB) design B $(K, \lambda; v)$ is a GD $(K, \lambda, \{1\}; v)$ of type 1^v. A set K of positive integers is said to be PBD-closed if $K = B(K)$, where $B(K) = \{ v$: a $B(K, 1; v)$ exists $\}$.

Corollary 2.4. Let NDB $(k, \lambda, F) = \{v$: an NDB $(k, \lambda; v)$ of form F exists$\}$. Then the NDB (k, λ, F) is a PBD-closed set.

Proof. For convenience of notation, let NDB denote the NDB (k, λ, F). Obviously, NDB $\subseteq$ B (NDB). We need only to show B (NDB) $\subseteq$ NDB. Let $v \in$ B (NDB), i.e., a B (NDB, 1; v) exists. Then Theorem 2.3 with $\omega(x) = 1$ for each treatment x shows that $v \in$ NDB, which completes the proof.

3. *Existence of* NDB $(3, \lambda; v)$'s

Now we prove the existence of an NB $(3, \lambda; v)$ in general. Note that only the case when the form is $(2, \lambda_2)^1$ needs to be investigated. First the case $\lambda_2 = 1$ is considered, which necessarily implies $\lambda = 3$.

Lemma 3.1. There exists an NDB $(3, 3; v)$ of form $(2, 1)^1$ for $v = 3, 4, 5, 6, 7, 8$.

Proof. These designs can be constructed as follows. Here the treatments underlined with "_" form the sub-blocks.

(1) NDB(3, 3; 3) of form $(2, 1)^1$: $\mathcal{V} = Z_3$,
$\mathcal{B} = \{ (\underline{1}, \underline{0}, 2), (\underline{1}, 0, \underline{2}), (1, \underline{0}, \underline{2},) (\underline{2}, \underline{0}, 1), (\underline{2}, 0, \underline{1}), (2, \underline{0}, \underline{1}) \}$.

(2) NDB(3, 3; 4) of form $(2, 1)^1$: $\mathcal{V} = Z_4$,
$\mathcal{B} = \{ (\underline{0}, \underline{2}, 1), (\underline{0}, 2, \underline{1}), (0, \underline{2}, \underline{1}) \bmod 4 \}$.

(3) NDB(3, 3; 5) of form $(2, 1)^1$: $\mathcal{V} = Z_5$,
$\mathcal{B} = \{(\underline{0}, \underline{1}, 2), (\underline{0}, \underline{2}, 4), (\underline{0}, \underline{3}, 1), (\underline{0}, \underline{4}, 3) \bmod 5 \}$.

(4) NDB(3, 3; 6) of form $(2, 1)^1$: $\mathcal{V} = Z_5 \cup \{ \infty \}$,
$\mathcal{B} = \{ (\underline{0}, \underline{\infty}, 4), (\underline{0}, \infty, \underline{4}), (0, \underline{\infty}, \underline{4}), (\underline{0}, \underline{1}, 3), (\underline{0}, 1, \underline{3}), (0, \underline{1}, \underline{3}) \bmod 5 \}$

(5) NDB(3, 3; 7) of form $(2, 1)^1$: $\mathcal{V} = Z_7$,
$\mathcal{B} = \{ (\underline{0}, \underline{1}, 3), (\underline{0}, 1, \underline{3}), (0, \underline{1}, \underline{3}), (\underline{0}, \underline{6}, 4), (\underline{0}, 6, \underline{4}), (0, \underline{6}, \underline{4}) \bmod 7 \}$.

(6) NDB(3, 3; 8) of form $(2, 1)^1$: $\mathcal{V} = Z_8$,
$\mathcal{B} = \{ (\underline{0}, 4, \underline{1}), (\underline{0}, \underline{4}, 2), (\underline{0}, 4, \underline{3}), (\underline{0}, 1, \underline{7}), (\underline{0}, \underline{2}, 1), (0, \underline{5}, \underline{3}), (\underline{0}, 2, \underline{5}) \bmod 8 \}$.

Theorem 3.2. There exists an NDB(3,3; v) of form $(2, 1)^1$ whenever $v \geq 3$.

Proof. It is already seen that NDB(3,3; v) of form $(2, 1)^1$ for $v = 3, 4, 5, 6, 7$ and 8. Hence assume $v \geq 9$. Now let $v = 3m, 3m + 1$ or $3m + 2$ for $m \geq 3$. When $m = 2n - 1$, define $L(i, j) = (i + j)/2$, where $i, j \in Z_{2n-1}$. When $m = 2n$, define $L(i, j) = (i + j)/2$, if $j \neq i + 1$, $L(i, i + 1) = \infty$, $L(i, \infty) = (2i + 1)/2$, $L(\infty, j) = (2j - 1)/2$ and $L(\infty, \infty) = \infty$, where $i, j \in Z_{2n-1}$. Now further let $\mathcal{V} = X \times Z_3$, where $X = Z_{2n-1}$ if $|X| = 2n - 1$ and $X = Z_{2n-1} \cup \{\infty\}$ if $|X| = 2n$. Then a collection $\mathcal{B}$ of blocks is formed as follows. For $x \in X$ and $i \in Z_3$ write (x, i) as x_i. Let $a, b \in X, a \neq b$. Add to $\mathcal{B}$ three blocks $(\underline{a_i}, \underline{L(a, b)_{i+1}}, b_i)$, $(\underline{a_i}, L(a, b)_{i+1}, \underline{b_i})$, and $(a_i, \underline{L(a, b)_{i+1}}, \underline{b_i})$ for each $i \in Z_3$. To complete $\mathcal{B}$ to be an NDB (3, 3; 3m) of form $(2, 1)^1$, on each $\{a\} \times Z_3$ for $a \in X$, place the blocks (and thus the sub-blocks) of an NDB(3,3; 3) of form $(2, 1)^1$. To obtain instead an NDB (3,3; 3m + 1) of form $(2, 1)^1$, add one further treatment ∞. Then for $a \in X$, on $(\{a\} \times Z_3) \cup \{\infty\}$, place the blocks (and thus the sub-blocks) of an NDB(3,3; 4) of form $(2, 1)^1$. To construct an NDB(3,3; 3m + 2), add two further treatments ∞_0 and ∞_1. For a single treatment $x \in X$, place the blocks (and thus the sub-blocks) of an NDB(3,3; 5) of form $(2, 1)^1$ on $(\{x\} \times Z_3) \cup \{\infty_0, \infty_1\}$. For all other treatments $a \in X, a \neq x$, include, for each block (a, b, c) below, three new blocks $(\underline{a}, \underline{b}, c)$, $(\underline{a}, b, \underline{c})$ and $(a, \underline{b}, \underline{c})$:

$$(\infty_0, a_0, a_1), (a_0, \infty_0, a_2), (a_1, a_2, \infty_0)$$

$$(\infty_1, a_2, a_1), (a_1, \infty_1, a_1), (a_2, a_0, \infty_1)$$

The resulting design is an NDB(3,3; 3m + 2) of form $(2, 1)^1$. Here the treatments underlined with "_" form the sub-blocks.

Theorem 3.3: The necessary and sufficient conditions for the existence of an NDB(3,λ; v) of form $(2, \lambda_2)^1$ are that $\lambda = 3\lambda_2$ and $v \geq 3$.

Proof. The necessity follows from Theorem 1.1. The sufficiency can be obtained by repeating every block (and thus every sub-block) of the designs in Theorem 3.2, λ_2 times.

Apart from this simple and direct construction, nested directed BIB designs can also be easily constructed by using the PBD-closure. A PB design $B(\{3, 4, 5, 6, 8\}, 1; v)$ exists whenever $v \geq 3$ (see, for example, Beth *et al.* [1], Section (IX.7.1. b), and an NDB$(3, 3\lambda_2; v)$ of form $(2, \lambda_2)^1$ for $v = 3, 4, 5, 6$ and 8 exists from Lemma 3.1 by repeating every block (and thus every sub-block) of the corresponding NDB(3,3; v) of form $(2, 1)^1$ λ_2 times. Then by applying Corollary 2.4, we can present an alternative proof of the existence of nested directed BIB designs NDB$(3, 3\lambda_2; v)$ of form $(2, \lambda_2)^1$.

4. *Existence of NDB(4, λ; v)'s*

We here prove the existence of an NDB$(4, \lambda; v)$. There are four forms, i.e., $(2, \lambda_2)^1, (2, \lambda_2)^2, (2^2, \lambda_2)^1$ and $(3, \lambda_3)^1$, to be considered.

4.1 Existence of NDB(4, λ; v)'s of forms $(2, \lambda_2)^1$ *and* $(2, \lambda_2)^2$

We start with an NDB$(4, \lambda; v)$ of form $(2, \lambda_2)^2$, since its existence implies immediately the existence of an NB $(4, \lambda; v)$ of form $(2, \lambda_2)^1$.

Theorem 4.1.1. The necessary and sufficient conditions for the existence of an NDB$(4, \lambda; v)$ of form $(2, \lambda_2)^2$ are that $\lambda = 6\lambda_2$ and $v \geq 4$.

Proof. The necessity follows from Theorem 1.1. We consider the sufficiency. For every $v \geq 4$, Kageyama and Miao [5] proved the existence of an NB $(4, 6\lambda_2; v)$ of form $(2, \lambda_2)^2$. For every block $\{[a, b], \langle c, d \rangle\}$ of the NB$(4, 6\lambda_2; v)$ of form $(2, \lambda_2)^2$, $(\mathcal{V}, \mathcal{B})$, where square and angle brackets indicate two distinguished systems of sub-blocks, define two new blocks $(\underline{a}, \underline{b}, \tilde{c}\ \tilde{d}), (\tilde{d}\ \tilde{c}, \underline{b}, \underline{a}) \in \mathcal{B}'$, where the treatments in a block underlined with "_" and "~" form two distinguished systems of sub-blocks. Then $(\mathcal{V}, \mathcal{B}')$ is an NDB $(4, 6\lambda_2; v)$ of form $(2, \lambda_2)^2$. This completes the proof of the sufficiency.

As an immediate result, we have the following.

Theorem 4.1.2. The necessary and sufficient conditions for the existence of an NDB $(4, \lambda; v)$ of form $(2, \lambda_2)^1$ are that $\lambda = 6\lambda_2$ and $v \geq 4$.

4.2. Existence of NDB (4, λ; v)'s of form $(2^2, \lambda_2)^1$

Some direct constructions for an NDB$(4, 3; v)$ of form $(2^2, 1)^1$ are first presented.

Lemma 4.2.1. Let $q \equiv 3 \bmod 4$ be a prime power. Then there exists an NDB (4, 3; q) of form $(2^2, 1)^1$.

Proof. Let $\mathcal{V}$ = GF (q) and $\mathcal{B}$ = {(θ^{t-1} , θ^{2t-1} , θ^t 1), (θ^i, θ^{t+i}, θ^{t+1+i}, θ^{1+i}), i = 0, 1, . . . , t − 2, mod q } where t = (q − 1)/2, θ is a primitive element of GF(q) and the treatments in any block underlined with "_" and with "=" form two sub-blocks. Then it follows that ($\mathcal{V}$, $\mathcal{B}$) is an NDB(4, 3; q) of form $((2^2, 1)^1$.

Lemma 4.2.2. There exists an NDB (4,3; v) of form $(2^2, 1)^1$ for v=6, 10, 14, 15, 18.

Proof. These designs can be constructed directly as follows. Here the treatments in a block underlined with "_" and with "=" form two sub-blocks.

(1) NDB(4, 3;6) of form $(2^2, 1)^1$: $\mathcal{V}$ + $Z_5 \cup \{\infty\}$, $\mathcal{B}$ = { (0, 1, 3, 2), (0, 3, 2, ∞), (∞, 0, 1, 4) mod 5}.

(2) NDB (4, 3; 10) of form $(2^2, 1)^1$: $\mathcal{V} = Z_9 \cup \{\infty\}$, $\mathcal{B}$ = {(0, 1, 3, 8), (0, 4, 1, 3), (0, 5, 3, 2,), (∞, 0, 4, 5), (0, 7, 4 ∞) mod 9 }.

(3) NDB (4, 3; 14) of form $(2^2, 1)^1$: $\mathcal{V} = Z_{13} \cup \{\infty\}$, $\mathcal{B}$ = { (0, 1, 3, 9), (9, 3, 1, 0), (0, 1, 3, 9), (9, 3, 1, 0), (1, 0, 3, 9), (9, 3, 1, ∞), (∞, 5, 2, 6) mod 13 }.

(4) NDB (4, 3; 15) of form $(2^2, 1)^1$: $\mathcal{V} = Z_3 \times Z_5$, $\mathcal{B}$ = {((1, 1),(1, 4) (2, 2), (2, 3)), ((2, 3), (2, 2), (1, 4), (1, 1)), ((1, 2), (1, 3), (2, 1), (2, 4)), ((2, 4), (2, 1), (1, 3), (1, 2)), ((0, 0), (0, 1), (1, 0), (2, 0)), ((2, 0), (1, 0), (0, 0), (0, 2)), ((2, 3), (1, 1), (1, 4), (2, 2)) mod (3, 5)}.

(5) NDB(4,3; 18) of form $(2^2, 1)^1$: $\mathcal{V} = Z_{17} \cup \{\infty\}$, $\mathcal{B}$ = {(1, 4, 13, 16), (16, 13, 4, 1), (3, 5, 12, 14), (14, 12, 5, 3), (2, 8, 9, 15), (15, 9, 8, 2), (6, 11, 10, 7), (∞, 16, 15, 11), (9, 10, 13, ∞) mod 17 }.

Lemma 4.2.3. There exists an NDB (4, 3; v) of form $(2^2, 1)^1$ whenever $v \equiv 0$ or 1 mod 4.

Proof. For each block {[a, b], [c, d]} of an NB (4, 3; v) of form $(2^2, 1)^1$, ($\mathcal{V}$, $\mathcal{B}$), define two new blocks (a, b, c, d), (d, c, b, a) ∈ $\mathcal{B}'$, where the treatments in a block underlined with "_" and with "=" form two sub-blocks. Then it follows that ($\mathcal{V}$, $\mathcal{B}'$) is an NDB(4,3; v) of form $(2^2, 1)^1$. On the other hand, an NB(4,3; v) of form $(2^2, 1)^1$ exists whenever $v \equiv 0$ or 1 mod 4 (see Kegeyama and Miao [5]). This completes the proof.

Lemma 4.2.4. There exists a B ({ 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23 }, 1;v) whenever $v \geq 4$.

Proof. See, for example, Beth *et al.* [1], Section (IX.7.1.e).

Now we can prove the main result of this sub-section.

Theorem 4.2.5. The necessary and sufficient conditions for the existence of an NDB $(4, \lambda; v)$ of form $(2^2, \lambda_2)^1$ are that $\lambda = 3\lambda_2$ and $v \geq 4$.

Proof. The necessity follows from Theorem 1.1. For the sufficiency, apply Corollary 2.4 with Lemma 4.2.4, where an NDB $(4, 3\lambda_2; v)$ of form $(2^2, \lambda_2)^1$ for v= 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19 and 23 can be constructed by repeating every block (and thus every sub-block) of the corresponding NDB(4,3; v) of form $(2^2, 1)^1$ in Lemmas 4.2.1, 4.2.2 and 4.2.3 λ_2 times.

4.3. Existence of NDB(4, λ; v)'s of form $(3, \lambda_3)^1$

In Kageyama and Miao [5], the authors showed that the necessary and sufficient conditions for the existence of an NB $(4, \lambda; v)$ of form $(3, \lambda_3)^1$ are that $\lambda = 2\lambda_3$ and $\lambda_3 (v-1) \equiv 0 \bmod 6$. This result can be utilized to construct some NDB $(4, \lambda; v)$ of form $(3, \lambda_3)^1$.

Lemma 4.3.1. There exists an NDB $(4, \lambda; v)$ of form $(3, \lambda_3)^1$ whenever $\lambda = 2\lambda_3$ and $\lambda_3 (v-1) \equiv 0 \bmod 6$.

Proof. For each block $\{[a, b, c], d\}$ of an NB $(4, \lambda; v)$ of form $(3, \lambda_3)^1$, $(\mathcal{V}, \mathcal{B})$, where [a, b, c] is a sub-block, form two new blocks $(\underline{a}, \underline{b}, \underline{c}, d), (d, \underline{c}, \underline{b}, \underline{a}) \in \mathcal{B}'$. Here the treatments in a block underlined with "_" form a sub-block. Then it follows that $(\mathcal{V}, \mathcal{B}')$ is an NDB $(4, \lambda; v)$ of form $(3, \lambda_3)^1$.

It can be seen from Theorem 1.1 that the necessary conditions for the existence of an NDB $(4, \lambda; v)$ of form $(3, \lambda_3)^1$ are that $\lambda = 2\lambda_3$ and $\lambda_3 (v-1) \equiv 0 \bmod 3$. First the case $\lambda_3 = 1$ is considered, which implies necessarily $\lambda = 2$.

Lemma 4.3.2. There exists an NDB(4, 2; 4) of form $(3, 1)^1$.

Proof. Let $\mathcal{V} = Z_4$ and $\mathcal{B} = \{(\underline{0}, \underline{1}, \underline{2}, 3), (\underline{1}, \underline{0}, \underline{3}, \underline{2}), (\underline{3}, \underline{2}, \underline{1}, 0), (\underline{2}, \underline{3}, \underline{0}, 1)\}$ where the treatments in a block underlined with "_" form a sub-block. Then $(\mathcal{V}, \mathcal{B})$ is an NDB(4, 2; 4) of form $(3, 1)^1$.

Lemma 4.3.3. There exists a $B(\{4, 7\}, 1; v)$ whenever $v \equiv 1 \bmod 3$ and $v \neq 10, 19$.

Proof. See, for example, Beth *et al.* [1], Section (IX.7.1.c).

Thus we can obtain the following.

Theorem 4.3.4. There exists an NDB(4, 2; v) of form $(3, 1)^1$ whenever $v \equiv 1 \bmod 3$ except possibly for v=10.

Proof. By applying Corollary 2.4 with Lemma 4.3.3, where an NDB(4, 2; 4) and an NDB(4, 2; 7) of form $(3, 1)^1$ exist from Lemmas 4.3.1 and 4.3.2, we can show the existence of an NDB(4, 2; v) of form $(3, 1)^1$ for every $v \equiv 1 \bmod 3$ except possibly for $v = 10, 19$. An NDB (4, 2; 19) of form $(3, 1)^1$ was already constructed in Lemma 4.3.1.

An immediate result of Lemma 4.3.1 is the following.

Theorem 4.3.5. There exists an NDB(4, 4; v) of form $(3, 2)^1$ whenever $v \equiv 1 \bmod 3$.

Now the case $\lambda_3 = 3$ is considered, which implies necessarily $\lambda = 6$.

Lemma 4.3.6. There exists an NDB(4, 6; v) of form $(3, 3)^1$ for v = 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23.

Proof. An NDB(4, 6; 4) of form $(3, 3)^1$ can be obtained by repeating every block (and thus every sub-block) of an NDB(4, 2; 4) of form $(3, 1)^1$ (Lemma 4.3.2) three times. Lemma 4.3.1 shows the existence of an NDB(4, 6; v) of form $(3, 3)^1$ for every odd v. The remaining designs can be constructed directly as the following shows. Here the treatments in a block underlined with "_" form a sub-block.

(1) NDB(4, 6; 6) of form $(3, 3)^1$: $\mathcal{V} = Z_5 \cup \{\infty\}$, $\mathcal{B} = \{(\underline{0}, \underline{\infty}, \underline{3}, 4), (\underline{0}, \underline{\infty}, \underline{1}, 3), (\underline{0}, \underline{\infty}, 1, \underline{3}), (\underline{0}, \underline{4}, \underline{3}, \infty), (\underline{0}, \underline{1}, \underline{2}, 4), (\underline{0}, \underline{2}, \underline{4}, 3) \bmod 5\}$.

(2) NDB(4, 6; 8) of form $(3, 3)^1$: $\mathcal{V} = Z_7 \cup \{\infty\}$, $\mathcal{B} = \{(\underline{0}, \underline{\infty}, \underline{1}, 2), (0, \infty, 1, 3), (2, 1, \underline{\infty}, \underline{0}), (\underline{3}, \underline{1}, \infty, \underline{0}), (\underline{0}, \underline{2}, \underline{4}, 6), (\underline{6}, 4, \underline{2}, \underline{0}), (\underline{0}, 3, \underline{6}, \underline{2}), (\underline{2}, 6, \underline{3}, \underline{0}) \bmod 7\}$.

(3) NDB(4, 6; 10) of form $(3, 3)^1$: $\mathcal{V} = Z_9 \cup \{\infty\}$, $\mathcal{B} = \{(\underline{1}, \underline{0}, \infty, \underline{2}), (2, \underline{\infty}, 0, 1), (0, \underline{\infty}, 3, 1), (1, 3, \underline{\infty}, 0), (0, 3, 6, 8), (8, \underline{6}, \underline{3}, \underline{0}), (\underline{0}, \underline{2}, \underline{4}, 6), (6, \underline{4}, \underline{2}, \underline{0}), (\underline{0}, \underline{4}, \underline{8}, 3), (3, \underline{8}, \underline{4}, \underline{0}) \bmod 9\}$.

(4) NDB (4, 6; 12) of form $(3,3)^1$: $\mathcal{V} = Z_{11} \cup \{\infty\}$, $\mathcal{B} = \{(0, \infty, 1, 2)$, $(2, 1, \infty, 0)$, $(0, \infty, 1, 3)$, $(3, 1, \infty, 0)$, $(0, 2, 4, 6)$, $(6, 4, 2, 0)$, $(0, 3, 6, 9)$, $(9, 6, 3, 0)$, $(0, 4, 8, 1)$, $(1, 8, 4, 0)$, $(0, 5, 10, 4)$, $(4, 10, 5, 0)$ mod $11\}$.

(5) NDB (4, 6; 14) of form $(3,3)^1$: $\mathcal{V} = Z_{13} \cup \{\infty\}$, $\mathcal{B} = \{(0, \infty, 1, 2)$, $(2, 1, \infty, 0)$, $(1, \infty, 0, 3)$, $(3, 0, \infty, 1)$, $(0, 2, 4, 6)$, $(6, 4, 2, 0)$, $(0, 3, 6, 9)$, $(9, 6, 3, 0)$, $(0, 4, 8, 12)$, $(12, 8, 4, 0)$, $(0, 5, 10, 2)$, $(2, 10, 5, 0)$, $(0, 6, 12, 5)$, $(5, 12, 6, 0)$ mod $13\}$.

(6) NDB (4, 6; 18) of form $(3,3)^1$: $\mathcal{V} = Z_{17} \cup \{\infty\}$, $\mathcal{B} = \{(0, \infty, 1, 2)$, $(2, 1, \infty, 0)$, $(1, \infty, 0, 3)$, $(3, 0, \infty, 1)$, $(0, 2, 4, 6)$, $(6, 4, 2, 0)$, $(0, 3, 6, 9)$, $(9, 6, 3, 0)$, $(0, 4, 8, 12)$, $(12, 8, 4, 0)$, $(0, 5, 10, 15)$, $(15, 10, 5, 0)$, $(0, 6, 12, 1)$, $(1, 12, 6, 0)$, $(0, 7, 14, 4)$, $(4, 14, 7, 0)$, $(0, 8, 16, 7)$, $(7, 16, 8, 0)$ mod $17\}$

Theorem 4.3.7. There exists an NDB(4, 6; v) of form $(3,3)^1$ whenever $v \geq 4$.

Proof. Apply Corollary 2.4 with Lemmas 4.2.4. and 4.3.6.

The following result can be easily proved.

Theorem 4.3.8. The necessary conditions for the existence of an NDB(4, λ; v) of form $(3, \lambda_3)^1$, i.e., $\lambda = 2\lambda_3$ and $\lambda_3 (v-1) \equiv 0 \bmod 3$, are also sufficient except possibly for an NDB(4, 2; 10) of form $(3,1)^1$.

Proof. The necessity follows from Theorem 1.1. For the sufficiency, note that if $(\mathcal{V}\,\mathcal{B}')$ is an NDB(4, α; v) of form $(3, \alpha_3)^1$ and $(\mathcal{V}, \mathcal{B}'')$ is an NDB (4, β; v) of form $(3, \beta_3)^1$, then $(\mathcal{V}, \mathcal{B}' \cup \mathcal{B}'')$ is an NDB $(4, \alpha+\beta; v)$ of form $(3, \alpha_3 + \beta_3)^1$. By using Theorems 4.3.4, 4.3.5 and 4.3.7, we can obtain the required designs.

5. *Conclusion*

In this paper the following main existence result is established.

Theorem 5.1. The necessary conditions for the existence of an NDB (k, λ; v) of any possible form are also sufficient for k=3 and 4 except possibly for an NDB(4, 2; 10) of form $(3,1)^1$.

While it is probably true that the only possible exception mentioned above is not really exception at all, after a great deal of valiant effort, the authors reluctantly leave this case for another day.

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